

The Analysis of Cosmic Ray Data

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1. Introduction

A statement of statistical belief not uncommon in cosmic ray work is: "you need five sigmas to convince me". This has some justification, in that the history of cosmic rays contains many instances when a source or effect is claimed but not subsequently substantiated. Frequently this has been due to incorrect application of some statistical technique, often a failure to account fully for the 'degrees of freedom'.

Most of the present body of statistical knowledge has been developed for specific problems, few of which occur in cosmic rays, although one of the most useful of texts[1] was produced for experimental particle physicists. The analyser of cosmic ray data has particular problems: cosmic ray data requires great effort in collection and they are unlikely, once analysed, to be repeated. The numbers are frequently small and there are usually data missing and frequently there is significant contamination by noise. Ideally, a statistical measure should be developed specifically for each application. This is the only way in which all of the parameters of the experiment can be allowed for in the analysis. It is more usual for a general statistical tool to be applied, for example χ^2 , which may not be optimal for the purpose, and for some experimental variables to be ignored.

The focus of this review will be on methods of determining the presence of a signal rather than estimating some parameters of the data. The aim is to gather together the recent developments in methods of analysis of the temporal and spatial features of cosmic ray data, especially where the methods used are not 'traditional'.

Several new methods have been published recently which depend on Bayesian ideas, and these ideas have been introduced before the description of the methods.

2. On/Off Counts

2.1. Introduction

The subject of detecting the presence of a source in counting rate data, using off-source control data has appeared many times[1, 2, 3, 4, 5]. Despite these numerous airings, erroneous statistical significances are occasionally still being published. In principle the question is easy to pose: if N_{ON} counts are detected when an instrument is pointed at a source and there is also a background counting rate, and N_{OFF} counts are detected when it is collecting background counts only under otherwise identical conditions, what is the likelihood that there is a genuine source?

A common treatment is to give for the significance of the excess counts:

$$N_{SIGMA} = \frac{N_{ON} - \alpha N_{OFF}}{\sqrt{N_{ON} + \alpha^2 N_{OFF}}} \quad (1)$$

where α is the ratio of time on-source to time off-source, $t_{ON} = \alpha t_{OFF}$. This is based on the supposition that the best estimate of the observed 'signal' is $N_{ON} - \alpha N_{OFF}$, the variances of the ON and OFF counts for a Poisson distribution are N_{ON} and N_{OFF} and the variance of the difference between N_{ON} and αN_{OFF} is the weighted

sum of the variances. The statistic used is *Student's t* which in the limit of large numbers is Gaussian. Since the distributions of N_{ON} and N_{OFF} are Poissonian, this expression should be used only if the numbers of events is sufficiently large for a Gaussian approximation to Poissonian to be valid.

It is an example of only one type of statistic which could be used in *ON/OFF* situations - a *goodness-of-fit* statistic to determine whether the observed data could have arisen from an *a priori* distribution. Other statistics could have been used, for example χ^2 , which in this instance would have one degree of freedom. Asymptotically they should have the same result, that is they both should reject or accept the null hypothesis equally. In these tests the *null hypothesis* is that the observations were both samples from the same population and that any difference arose merely by chance. There is no explicit alternative hypothesis, but an implicit one: that if the difference between the counts was unlikely to be due to chance, it arose because of a genuine source, strength unspecified.

2.2. Likelihood Analysis

An optimal test exists for the intermediate case where there are two completely specified hypotheses: H_0 : the *null hypothesis* as described above, and H_1 : a hypothesis involving another model, usually including a specific 'signal'. In this rare (in cosmic rays) case, the Neyman/Pearson theorem shows that the likelihood ratio is optimal for any distribution function for the errors.

In the more usual case, H_1 is not fully specified, but has one or more free parameters. The null hypothesis H_0 is that N_{OFF} and N_{ON} are both samples of the same population for which the source strength $S = 0$. The alternative hypothesis H_1 is that N_{ON} contains an *unknown* source component, $S > 0$. In this case there is no optimal test, except that for errors of the exponential family, such as a Gaussian, the likelihood ratio is expected to be near-optimal.

The problem was discussed at length twenty five years ago by O'Mongain[3] and Hearn[2] but was not solved satisfactorily, at least in this field, until the maximum likelihood treatment of Gibson et al. [4] and Dowthwaite et al.[6] and later by a similar treatment by Li and Ma[5]. In these treatments the observed *ON* and *OFF* counts are due to (i) an unknown background B plus an unknown source S and (ii) the same unknown background B alone. The likelihood ratio is maximised with respect to the possible source counts:

$$\begin{aligned} \lambda &= \left(\frac{P(N_{ON}, N_{OFF} | S = 0)}{P(N_{ON}, N_{OFF} | S = N_{ON} - \alpha N_{OFF})} \right) \\ &= \left[\frac{\alpha}{1 + \alpha} \left(\frac{N_{ON} + N_{OFF}}{N_{ON}} \right) \right]^{N_{ON}} \left[\frac{1}{1 + \alpha} \left(\frac{N_{ON} + N_{OFF}}{N_{OFF}} \right) \right]^{N_{OFF}} \end{aligned} \quad (2)$$

A standard result[1] is that the probability of obtaining a given λ is obtained from

$$-2 \ln(\lambda) \sim \chi^2(1) \quad (3)$$

2.3. Comparison of methods

Both equations 1 and 3 are valid asymptotically: only for large values of N_{OFF} and N_{ON} . Equation 1 assumes that the error distributions of N_{ON} and N_{OFF} are gaussian and equation 3 assumes that $-2\log(\lambda)$ is distributed as $\chi_2(1)$. To check the region of validity, random data sets have been generated for each of a number of values of N_{OFF} . For each data set α has been set to 1.0 and a value of $S_3 = 3\sqrt{2B + S_3}$ has been calculated, which is the 3σ value of $S = S_3$ assuming the validity of equation 1. At each value of N_{OFF} , 10^6 data sets were generated using a poissonian random number algorithm[7]. The fraction of samples N of N_{OFF} where $N > N_{OFF} + S_3$ is used as an estimate of the true probability of obtaining $N_{ON} = N_{OFF} + S_3$ by chance. The results are shown in figure 1 where both equations 1 and 3 are shown to overestimate the probability near the 3σ level almost equally likely, the former slightly less so. It is evident that, near the 3σ level, there is little to choose and both equations are adequate for values of N_{OFF} and N_{ON} of a few hundred or more. Since good algorithms are available for Poissonian random number generation it is likely to be better to determine the probability of N_{ON} and N_{OFF} for values less than ~ 100 using Monte Carlo methods tailored for the exact values of N_{ON} and N_{OFF} .

3. Time Series

3.1. Introduction

Time series analysis has been the subject of very many books and articles and has been applied in very many fields. The term covers a wide range of concepts, including Change Point Analysis, Fourier Analysis and Trend Analysis. In cosmic ray studies, there are several areas of application, such as sidereal/solar effects on low energy cosmic rays on the ground, periodicity in data from point sources, either from satellite X- and γ -ray

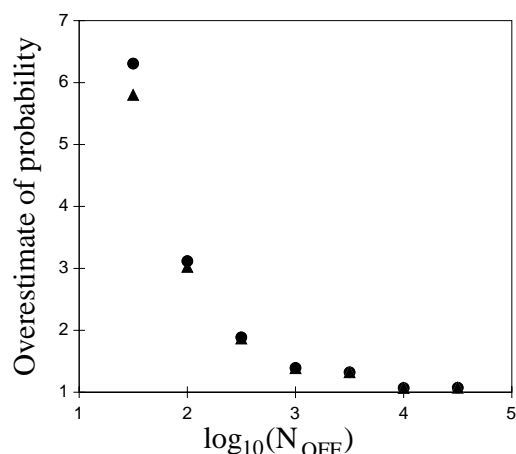


Figure 1. Ratio of probabilities from equation 1 (full circles) and equation 2 (full triangles) to Monte Carlo results for various values of N_{OFF} and $N_{ON} = N_{OFF} + S_3$

data, or from ground-based Čerenkov detectors, and sporadic emission of a wide range of cosmic ray energies. In these cases, the raw data is usually in the form of time-tagged events.

3.2. Bursts of Events

This section will be concerned with the problem of deciding whether the counting rate of a detector has deviated from the expected rate due to a real outburst of events. The problem is usually most difficult in data comprising time-tagged events. An initial analysis could start with binning the data and looking for a deviation from the expected Poissonian distribution of the counts. One problem with this approach is that in the model of a single Poisson process generating the counts, each bin is independent, the experimenter often has the freedom to place the bins, both in position and width, arbitrarily. This alters the 'degrees of freedom' and experience suggests that more bursts have been 'detected' in the past than could have been justified from the data.

The problem mentioned above is a specific one but in general most statistical problems associated with sporadic emission relate to the lack of a specific model for the form, duration and amplitude of emission, and the feeling is often that, given a free hand with the parameters, any pure noise series could be made to disclose a 'burst'. A recent paper by Scargle[8] suggests that existing methods for searching for rapid variability in X -ray and γ -ray astronomy do not fully extract all of the information contained in photon counts. The reasons given included 'binning fallacies', in that the data were widely binned and the size of the bins must be large enough to give 'good statistics'. Further, global methods such as autocorrelations and power spectra used on large data sets dilute the effects of sporadic bursts. The Bayesian response to these problems is discussed later.

The problem at first sight does not seem insolvable using classical statistical theory. The statistical treatment of *point processes*: data occurring as points on the real line, or as discrete times, is covered by several texts, for example Cox and Isham[9]. The general treatment covers a variety of statistical processes, including Poisson (which is of most application here), doubly stochastic Poisson (where the average Poisson rate is itself a variable) and renewal processes where the distribution function for intervals between points is not exponential. In analysing data in the form of time-tagged photons without appreciable dead time, classical statistics would look for a powerful goodness-of-fit test of the pure Poisson process, if possible avoiding the loss of information and the arbitrary choices associated with binning.

Given such a series of times, the problem posed here is: is there evidence for 'bunching' or 'bursts'? Alternatively, are the data consistent with a uniform distribution in time which, for events not affected by counter dead-time, would be governed by a pure Poisson process? Some recent papers such as McLaughlin et al.[10] use just this assumption to classify sources into 'steady' or 'variable'. Others use *ad hoc* methods to estimate the probability of bursts[11].

3.2.1. The Scan Statistic The test statistic postulated above exists: the Scan Statistic has been extensively studied by Parzen[12], Barton and David[13], Huntington and Naus[14], Neff and Naus[15], Naus[16], Glaz[17], Wallenstein, Naus and Glaz[18, 19], Chen and Glaz[20] and Månsson [21]. It is a statistic for detecting clustering in time or one dimension in space. It is usually described as the maximum (or minimum) number of events which can be found in a window of fixed duration scanning smoothly through a much longer interval containing discrete events following some random process, for example Poissonian. An example of the scan statistic is shown in figure 1 for window lengths of 1% and 10% of the duration of the data. The random test data has a constant mean rate except for the third quarter which has double the rate.

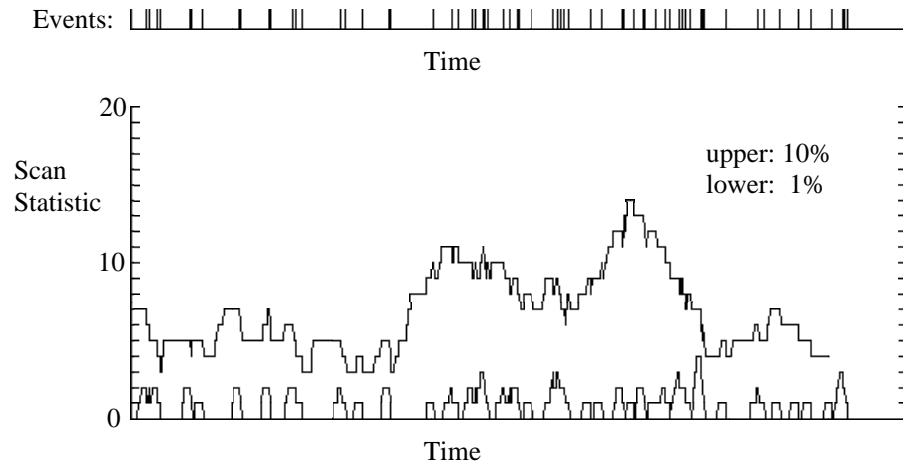


Figure 2. Example of the Scan Statistic for windows of 1% and 10% of the data duration

The scan statistic S has a probability $P(S)$ which depends on the rate of events, the duration and the width of the scanning window. Some exact solutions for the probability $P(S)$ have been provided. One of them, by Huntington and Naus[14], provides the probability of a related statistic:

$$P(a_n \leq a) = 1 - \sum_Q R \det |1/h_{ij}| \det |1/l_{ij}| \quad (4)$$

where a_n is the smallest interval a containing n events in the range $[0, 1]$, where this range contains N events in all. The summation extends over the set Q of all partitions of N into $2L+1$ integers satisfying $n_i + n_{i+1} < n, i = 1, \dots, 2L$ and $R = N!b^M(a-b)^{N-M}$ with $M = \sum_{k=0}^L n_{2k+1}$, and

$$\begin{aligned} h_{ij} &= \sum_{k=2j-1}^{2i-1} n_k - (i-j)n \quad L+1 \geq i \geq j \geq 1 \\ &= - \sum_{k=2i}^{2j-2} n_k + (j-i)n \quad 1 \leq i < j \leq L+1 \\ l_{ij} &= \sum_{k=2j}^{2i} n_k - (i-j)n \quad L \geq i \geq j \geq 1 \end{aligned}$$

n	exact	Newell-Ikeda
5	0.711	0.984
6	0.225	0.565
7	0.0425	0.129
8	6.31×10^{-3}	0.0196
9	8.04×10^{-4}	2.48×10^{-3}
10	9.05×10^{-5}	2.76×10^{-4}

Table 1. Comparison of Scan Statistic probabilities

$$= - \sum_{k=2i+1}^{2j-1} n_k + (j-i)n \quad 1 \leq i < j \leq L$$

Equation 4, although exact, is computationally expensive for large N and small a , that is a large data set with a small scanning window, but several approximations have been provided which are designed to be valid for certain combinations of parameters.

3.2.2. Newell-Ikeda Approximation for the Scan Statistic The Newell-Ikeda[22, 23] asymptotic formula is suitable for small probabilities. It gives the probability of finding a section of length t in a data set of length T , given a Poisson process of average rate λ :

$$P(n; \lambda T, t/T) \doteq 1 - \exp(-\lambda^n t^{n-1} T / (n-1)!) \quad (5)$$

As shown in table 1, it significantly overestimates larger probabilities. Better approximations, although not as easy to calculate, are available, for example Conover, Bement and Iman [24] and Naus[25].

3.2.3. Naus Approximation for the Scan Statistic The more exact treatment of Naus[25] will be given without derivation. For an average rate of events λ , data of total duration T and scanning window of duration t , define $L = T/t$. Then the probability that the number of events in a scanning window never exceeds n is $Q^*(n; \lambda L, 1/L)$ and is accurately approximated by:

$$Q^*(n; \lambda L, 1/L) \doteq Q^*\left(n; 2\lambda, \frac{1}{2}\right) \left[Q^*\left(n; 3\lambda, \frac{1}{3}\right) / Q^*\left(n; 2\lambda, \frac{1}{2}\right) \right]^{L-2} \quad (6)$$

Note that this approximation is valid for a wide range of types of distribution for the time between events. Exact formulae for $Q^*(n; 2\lambda, \frac{1}{2})$ and $Q^*(n; 3\lambda, \frac{1}{3})$ are given for a Poisson process:

$$\begin{aligned} Q^*\left(n; 2\lambda, \frac{1}{2}\right) &= F_{n-1}^2 - (n-1)p_n p_{n-2} - (n-1-\lambda)p_n F_{n-3} \\ Q^*\left(n; 3\lambda, \frac{1}{3}\right) &= F_{n-3}^2 - A_1 + A_2 + A_3 - A_4 \end{aligned}$$

where

$$\begin{aligned}
A_1 &= 2p_n F_{n-1} ((n-1) F_{n-2} - \lambda F_{n-3}) \\
A_2 &= 0.5p_n^2 ((n-1)(n-2) F_{n-3} - 2(n-2) \lambda F_{n-4} + \lambda^2 F_{n-5}) \\
A_3 &= \sum_{r=1}^{n-1} p_{2n-r} F_{r-1}^2 \\
A_4 &= \sum_{r=2}^{n-1} p_{2n-r} p_r ((r-1) F_{r-2} - \lambda F_{r-3})
\end{aligned}$$

and p_i, F_n are the Poisson probability and distribution functions: $p_i = e^{-\lambda t} (\lambda t)^i / i!$ and $F_n = \sum_{i=0}^n p_i$.

Tight bounds for $Q^*(n)$ have been given by Glaz and Naus[26] and a recursive method proposed[27] for calculating $Q^*(n; 2t)$ and $Q^*(n; 3t)$ for situations where the random quantity X_i may take on values other than 0, 1, that is situations where an 'event' cannot be given as either present or absent but only with a non-zero probability.

Other approximations for the tail of the scan statistics and the moments of its distribution have been given by Glaz [17] and Chen and Glaz[20]. Sample tables of the scan statistic have been given for $n \leq 500$ by Glaz[28, 17].

This treatment of the scan statistic is for an interval of length t , specified in advance. When searching for a 'burst' of events, an *a priori* length cannot always be specified. An extension to the treatment above has been described by Nargawalla[29] in which the length need not be pre-assigned.

3.2.4. Alm Approximation for the Scan Statistic A new approximation has been given recently by Alm[30] which is accurate and easy to calculate for large values of T/t and λt . This treatment examines the distribution of upcrossings, that is occurrences where the number of events in the scanning window increases by 1 as the window is moved. By separating these events into *primary* and *secondary* upcrossings, the dependence of the second type from the first (almost) independent events allows significant simplifications. If each window of length t were independent, the expected number of events would be λt with a Poisson probability function $F_{\lambda t}(n)$ and distribution function $p_{\lambda t}(n)$. The approximation based on the ideas above gives the simple modification:

$$P(N \geq n) = 1 - F_{\lambda t}(n) \exp \left[- \left(1 - \frac{\lambda t}{n+1} \right) \lambda (T-t) p_{\lambda t}(n) \right] \quad (7)$$

Equation 7 has been tested for $\lambda t = 40$ and $T/t = 3600$ using 10000 Monte Carlo simulations. The results are shown in table 2 for $13 \leq n \leq 23$. It can be seen that equation 7 is a good approximation within the sampling errors.

3.2.5. Other Approximations for the Scan Statistic Other methods have been published, for example the Burst Expectation Search by Giles[31] and CUSUM by VanStekelenborg and Petrakis[32]. The first follows earlier work[33] which used binned

n	Monte Carlo	Equation 7
13	0.9984	0.9993
15	0.9429	0.9478
16	0.6654	0.6635
17	0.3094	0.3087
18	0.1055	0.1098
19	0.0307	0.0337
20	0.0093	0.0095
21	0.0021	0.0025
22	0.0006	0.0006
23	0.0002	0.0001

Table 2. Comparison of 1-D Scan Statistic probabilities for 10000 samples with $T/t = 3600$ and $\lambda t = 40$

times of events and calculates Poisson probabilities of bin counts from a running average of a sample of bins. The BES inverts this process and, for each possible bin count from zero to several hundreds, calculates the mean rate below which the possible count could be a significant burst using a fixed sample of bins around the trial bin. The aim of keeping a fixed sample was to avoid problems arising from a step function edge entering a moving average.

3.2.6. Bursts: Summary In summary, of possible methods suggested for searching for bursts using classical statistics, the Scan Statistic is recommended, both for time-tagged data and for time-binned data. For small data sets or large window sizes, equation 4 provides an exact probability of the largest number in any window arising due to chance. Many approximate formulae are available, depending on whether the probability of the scan statistic is expected to be large or small. In most practical cases in cosmic rays the statistic is used to search for a possible outburst and so the probability of a given value of the scan statistic arising due to chance will be small in order to be useful. It becomes a matter of computational convenience which of the formulae above is used but equation 7 delivers a good approximation over a wide range of probability values and is easy to calculate. It also has the advantage, in terms of understanding the principles, of starting from the naive initial Poissonian formula with non-overlapping (independent) windows. Its use is therefore recommended here.

3.3. Periodicity

Most of the methods for time-series analysis, including trend analysis and auto-regressive moving average (ARMA), have been developed for fields other than cosmic rays, for example [34, 35, 36]. Fourier methods suitable for data at equally spaced times are well developed but are usually not suitable, although these have been extended to discuss

unequal intervals and missing data[37]. A bibliography of astronomical time series analysis has been given by Koen[38].

3.3.1. The Rayleigh test and Dependants The spur for the introduction of the Rayleigh test into γ -ray astronomy was the unsatisfactory nature of the statistics being used before. Early tests on γ -ray data used epoch-folding to produce a histogram in phase, and χ^2 as a statistic for goodness of fit to a uniform distribution. This suffers from several disadvantages:

- (i) the freedom to select the number of bins,
- (ii) the freedom to define the starting phase,
- (iii) the failure to use the information contained in the order of the bins.

This last problem can be overcome to some degree by using the Run Test, which is independent and therefore whose probability may be combined with that from χ^2 . The result of the freedoms listed above is that different authors could return quite different chance probabilities, given the same data, despite using the same test statistic. The analysis of γ -ray data from the Crab pulsar by Gibson et al.[39] contained the first known use of the Rayleigh statistic[40, 41] in cosmic ray work. It is still a goodness-of-fit statistic, which has no explicit hypothesis as an alternative to the null hypothesis.

The time of each event is treated as a unit vector in the plane, with an angle equal to the pulsar phase. If N unit vectors of random orientations (random phases) are added, the distribution of the resultant R may be obtained from the distribution of the orthogonal components of the vectors, $\sin \phi_i$, $\cos \phi_i$ where ϕ_i is the phase of the i^{th} vector. The means of these components are :

$$C = \frac{1}{N} \sum_{i=1}^N \cos \phi_i \quad S = \frac{1}{N} \sum_{i=1}^N \sin \phi_i$$

From the Central Limit Theorem (CLT) means of samples of C are distributed, for large N , as a Gaussian with $var_C = \sigma_C^2/N$. For vectors uniformly covering the circle:

$$\sigma_C^2 = \int_0^{2\pi} \cos^2(\phi) d\phi / 2\pi = 0.5$$

therefore $var_C = var_S = 1/2N$. The quantities C and S are asymptotically uncorrelated and have zero means. The statistic $2NR^2 = 2NC^2 + 2NS^2$ is therefore the sum of the squares of two zero-mean, unit-variance uncorrelated variables and is distributed as χ^2 with 2 degrees of freedom [42, 43]. The probability distribution function (pdf) of R is :

$$f(R)dR = 2NR e^{-NR^2} dR \quad (8)$$

and its cumulative probability distribution is :

$$F(R) = e^{-NR^2} \quad (9)$$

The quantity NR^2 is known as the Rayleigh power.

If a data set spans a time interval T the number of independent frequency trials in the frequency range f_1 to f_2 is $\nu = T(f_1 - f_2)$ if $T \gg 1/f_1, 1/f_2$, with the independent

frequencies separated by $1/T$. In practice allowance must be made for leakage: the possible effect of a signal at frequency f_0 on trial frequencies f with $|f - f_0| \gtrsim 1/T$, and oversampling: the possibility of obtaining a larger value of NR^2 by varying the frequency between adjacent independent frequencies. This has been done using by de Jager et al.[44] by Monte Carlo techniques and analytically by Orford[45]. Both methods agree that the number of trials is $nT(f_1 - f_2)$ where n is a slowly varying function of $F(R)$ in the range 2 to 4, with a value approximately 3 for $F(R) \sim 10^{-3}$.

3.3.2. The Z_n^2 Test The Z_n^2 test is the extension of the Rayleigh test to include harmonics. If n separate harmonics are included with independent coefficients, the statistic is

$$Z_n^2 = 2N \sum_{i=1}^n R^2(i\omega) \quad (10)$$

where $2NR^2(i\omega)$ is the Rayleigh power for the i^{th} harmonic. Z_n^2 is distributed as $\chi^2(2n)$. Variations on this technique depend on the method used to select the number and weighting of harmonics. A similar principle is used in radio astronomy where a pulse of width W is searched for using $P/2W$ harmonics which improves the signal to noise by a factor of up to $(P/2W)^{0.5}$ [46]. A search for γ -ray emission from radio pulsars proposed the use of Z_2^2 as a relatively powerful but general test for periodicity[47]. The power of the Rayleigh test for light curves from sinusoids to δ -functions was explored by Protheroe [48]. A variant of Z_n^2 is the H-test[44] in which the value of n is obtained objectively from the data and Z_n^2 is suitably rescaled. This last test is most suitable for multi-mode light curves.

3.3.3. Limitations of the Rayleigh and associated statistics The foregoing results for the Rayleigh (Z_1^2) and $Z_{n>1}^2$ tests are for the asymptotic case, that is: uncorrelated C and S with zero means. In most practical applications, these conditions are not strictly met. Ground-based gamma-ray observations of long-period pulsars are limited by:

- (i) being only a few hours in duration and
- (ii) variations in zenith angle, producing changing counting rates.

The requirement for large sample size is usually met - typical counting rates are ≈ 1 per second over several hours. The result is an enhancement of χ^2 in pure noise data for longer test periods - red noise. The first limitation listed above may be overcome by truncating the dataset so that only integral multiples of the trial period are tested - see Carramiñana et al.[50] and Raubenheimer and Ögelman[51]. As a result, the two trigonometric terms have zero expectations, given a constant mean counting rate. This truncation is easy to accomplish, but results in a variable data selection depending on the test period and therefore all periods are not accorded the same treatment. Since the periodogram is the convolution of the power spectral density with the Fourier transform of the data window, any spectral estimate based on a truncated data set is

biased[52, 42, 53]. Further, any correlation introduced by the second limitation above will not be removed this way. An attempt to remove the results of the counting rate variation has been made by Raubenheimer et al.[54] by fitting a parameter a in an *ad hoc* modification of the Rayleigh probability distribution:

$$F(R) = e^{-2aNR^2} \quad (11)$$

to random data sets containing no signal, but with the same parameters as the test data set. For data taken on Vela X-1 (period ≈ 5 minutes) they found that equation 11 with $a = 0.4$ (as opposed to 0.5 from simple theory) gave a probability distribution which was a good fit to the distribution in χ^2 for noise at periods near to 5 minutes in simulated data sets.

3.3.4. Modified Rayleigh Statistic If the expectations of C and S , their variances and their covariance are not assumed to be zero, $\frac{1}{2N}$ and zero respectively, but are calculated for a specific dataset, then the asymptotic probability equation 9 may be valid, given a sufficiently large number of events[55].

The expression for χ^2 in the case of samples of two correlated variables C and S is :

$$\chi^2 = \begin{bmatrix} \bar{C} - E(C) \\ \bar{S} - E(S) \end{bmatrix}^T \begin{bmatrix} \sigma_C^2 & cov_{S,C} \\ cov_{S,C} & \sigma_S^2 \end{bmatrix}^{-1} \begin{bmatrix} \bar{C} - E(C) \\ \bar{S} - E(S) \end{bmatrix} \quad (12)$$

For any data set, the substitution of the actual values of $E(C)$, $E(S)$, σ_C , σ_S and $cov_{S,C}$ will result in a value of χ^2 corrected for the correlation of the variables C and S and with a probability distribution, for large sample size, given by $\exp(-\chi^2/2)$.

In the case of a box-car data set with a constant average counting rate, a starting time t_1 and ending time t_2 with $T = t_2 - t_1$ and a trial period $P = 2\pi/\omega$:

$$\begin{aligned} E(C) &= \frac{[\sin \omega t]_{t_1}^{t_2}}{\omega T} \\ E(S) &= \frac{[-\cos \omega t]_{t_1}^{t_2}}{\omega T} \\ Nvar_C &= \frac{1}{2} + \frac{[\sin \omega t \cos \omega t]_{t_1}^{t_2}}{2\omega T} - [E(C)]^2 \\ Nvar_S &= \frac{1}{2} - \frac{[\sin \omega t \cos \omega t]_{t_1}^{t_2}}{2\omega T} - [E(S)]^2 \\ Ncov_{S,C} &= \frac{[\sin^2 \omega t]_{t_1}^{t_2}}{2\omega T} - E(C)E(S) \end{aligned}$$

These depend solely on ω , t_1 and t_2 and their substitution in equation 12 gives a χ^2 value corrected for the finite length of the data set. If it is known that there is no secular change in counting rate the substitution of the above equations into equation 12 would give the correct formal probability of chance occurrence, even if the duration of the data set is less than the trial period, as long as the number of events was high enough for the CLT to be valid. It is more usually the case in ground-based gamma-ray observations that the

box-car function is only an approximation. Monte Carlo simulations of data sets have been carried out to test the validity of data set truncation and the above formulations for the case of secular variations of counting rate superimposed on noise. In order to test the validity of the probability distribution equation 9 down to probabilities $\approx 10^{-6}$, data sets were generated using a multiplicative congruential algorithm with shuffling, chosen to avoid serial correlations. The repeat period is longer than 2×10^{18} . The time of each event t_i was generated from the previous event: $t_i = t_{i-1} - \Delta(t) \ln(rnd)$ where $\Delta(t)$ is the mean separation of events as a function of time. A group of 10^6 data sets of duration 8000 s were simulated with a counting rate profile $R = R_0(1 - 0.3t/4000)$ and $R_0 = 1s^{-1}$. Each data set was tested for periodicity at a trial period of 295s by finding C and S with reference to the time of the first event. These values were substituted into equation 12 for various assumptions about the form of $E(C)$, $E(S)$, var_C , var_S and $cov_{S,C}$. The probability of chance occurrence was calculated from $e^{-\chi^2/2}$.

The resulting cumulative frequency distributions for $e^{-\chi^2/2}$ have been calculated for the cases of (a) a truncation of the data set to integral multiple of the trial period, (b) box-car function and (c) a linear fit to the counting rate profile. The ratios of the observed to expected frequencies of occurrence of χ^2 chance probabilities is shown in figure 3, as functions of $\log(\chi^2 \text{ probability})$. Note that the duration of the data set is corrected for equally well by (b) and (c).

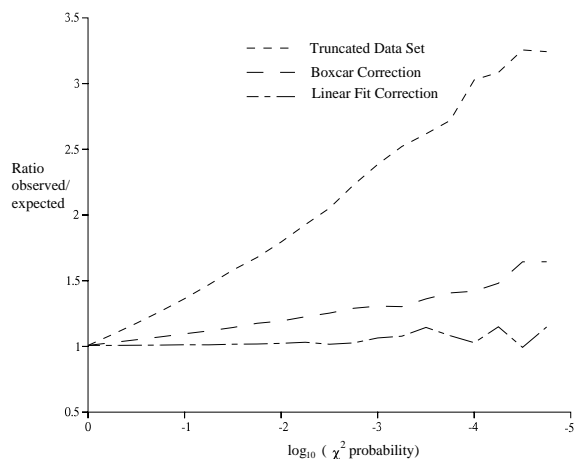


Figure 3. Ratio of cumulative frequency of χ^2 probabilities to expectation

The boxcar and truncated statistics both make corrections for the finite length of a data set, but give a residue which may be identified as being caused by the change of counting rate during the trial period. Longer trial periods or greater rates of change in counting rate would amplify their biases. The truncation method has a distribution which may be fitted, for this simulated data set, by a form such as equation 11 with $a = 0.45$. The linear fit model is seen to be a good representation of the noise spectrum down to chance probabilities of $\approx 10^{-5}$.

3.3.5. Other Tests Leahy et al.[56] pointed out that the unmodified Rayleigh test was powerful for detecting wide peaks in a light curve, in fact it is identical with a likelihood ratio test of a sinewave plus uniform against a uniform phase distribution [57]. In addition, for a light curve of a von Mises form (the circular generalisation of the Gaussian), the Rayleigh statistic exhausts the data's information on periodicity if the concentration parameter κ is allowed to vary freely [58].

Narrow periodic pulse detection, with significant power in the higher harmonics, is bound to be quite difficult because the number of degrees of freedom increases with the trial frequency range. Protheroe[59] proposed a test statistic

$$T_n = \frac{2}{n(n-1)} \sum_{i=1}^{n-1} \sum_{j=i+1}^n (\Delta_{ij} + 1/n)^{-1} \quad (13)$$

which looked for close clustering of points on the circle. In this statistic Δ_{ij} is the distance between the angles x_i and x_j of two events on the circle:

$$\Delta_{ij} = 0.5 - |[(x_i - x_j) \bmod 2\pi] - \pi|$$

The null distribution was found using Monte Carlo methods for $n \leq 200$ and critical values given. The context of the test was the search for ultra-high energy γ -rays from Cygnus X-3 and was therefore not designed for large n . In this limitation it is similar to the exact expression scan statistic described above. Others have suggested variants which are designed to be powerful for certain classes of pulsed emission [61].

The Scan Statistic may also be used for searching for non-uniformity in phase. For narrow windows its probability distribution is well approximated by the scan statistic on the line[60]. No systematic work on the use of the Scan Statistic in periodic analysis has been traced.

3.3.6. Searching for a Periodicity It is usually only the case that a unique periodic ephemeris is available for high energy photons from an isolated radio pulsar. In other cases, a search must be made in period, and the test used must allow for the freedoms associated with the trial period range. A rule-of-thumb arising from the number of 'degrees of freedom' implicit in a periodicity search using the Rayleigh test[44, 45] is that the search should be at intervals in period of $(IFI)/3$ (IFI = Independent Fourier Interval). For a period of P in a data set of duration T this corresponds to a trial period step of $\sim P^2/3T$. This step size has the advantage that the number of degrees of freedom to be used to interpret the peak periodic amplitude found is approximately the number of periods tried. If harmonics of the test period are to be included, the spacing would be correspondingly reduced and hence the number of trial periods increased. For the Z_n^2 test, the reduction in period step, and consequent increase in both computation and the degrees of freedom to be accounted for, is by a factor n .

When searching for pulsed emission from some sources, in particular binary sources, there is frequently poor knowledge of the both the pulsar period and period derivative. In this case the light curve will be narrow only if the correct period P and period

derivative \dot{P} is offered to the test. A nearby, but not correct, trial period and the ignorance of a period derivative will smear the light curve. If a true light curve were a δ function at period $P = P_o$ and the trial period was $P = P_o + \Delta$, the light curve would be a rectangular distribution in phase of length $T\Delta/P_o^2$. This effectively limits the number of harmonics which may be realistically added to $P_o^2/T\Delta - 1$.

Some searches for periodicity are combined with a search for a DC excess. This is common in Čerenkov telescope searches where ON-source data is compared with OFF-source control data to detect any DC component. The combined analysis of this situation was proposed by Lewis[63] in which a statistic α is defined as the sum of the Rayleigh statistic and the square of equation 1, distributed as $\chi^2(3)$. The assumption in this case is that all of the excess is pulsed; if there is an unpulsed component the test statistic will be biased. Again, the presence of a possible unpulsed component could be built into a Bayesian analysis.

3.3.7. Conclusion on Periodicity The question of the best classical test for the presence of periodicity is a complex one. The selection of the most sensitive test requires a knowledge of

- the pulsar ephemeris
- the light curve shape
- the background noise distribution

If all of these are known in advance, a most powerful test, based on the Z_n^2 extension of the Rayleigh test is likely to be close to optimum. Frequently some or all of these will be unknown or poorly known. In this case, some allowance must be made for the lack of knowledge and the test selected should not contain any assumption which causes a significant bias. It has sometimes been claimed that the Rayleigh test is 'biased' towards broad light curves and that a test which is more sensitive to narrow light curves should be used when such a light curve is suspected. This raises the problem, discussed in the previous section, of the smearing of a light curve if the pulsar's ephemeris is uncertain.

Protheroe has suggested[62] that if one has no information about the nature of the phase distribution one should be conservative and adopt the Rayleigh test. A rationale for this is that if one is searching for an unknown period and an unknown light curve, which is quite common in γ -ray work, and there is no significant power in the fundamental, then a test involving the addition of an unknown number of higher harmonics is unlikely to be successful. This point will be revisited later in discussing a Bayesian method of searching for periodicity.

A simple suggestion made before and reiterated here is that if Z_n^2 does not show evidence for periodicity, that is: there is no significant power in the fundamental or the first harmonic, either in addition or separately, then it is unlikely that the data will contain a strong periodic signal.

4. Spatial Analysis

4.1. Introduction

Spatial analysis of arrival direction data is of great interest for X- and γ -rays from satellites and for cosmic rays of the highest energies, which may not be greatly deflected in the galactic magnetic field.

Simple methods rely on a grid placed on the events and counts in the grid cells taken as independent Poisson-distributed events. If the cells are fixed absolutely, there is no problem in ascribing a suitable Poissonian probability to the largest number detected in any cell. If there is freedom to incrementally move the cell containing the largest count, a larger number is generally found. In this case the new cells created are correlated and the assumption of independence is incorrect: simple application of Poissonian probabilities is inappropriate. The problem of having the freedom to move the boundaries of the cells was pointed out for cosmic ray 'sources' by Hillas[65] who suggested a conservative number of 'sigmas'. Large scale anisotropy in gamma ray bursts were sought using dipole and quadrapole analysis[66]. A 'pair matching' statistic was used by Bennett and Rhie[67] to check for gamma ray burst repeaters rather than 'nearest neighbour' methods used by others[68] and criticised by Nowak[69]. Many methods have been used which are based upon a known point-spread function (PSF). Amongst these are Maximum Entropy methods such as those used for satellite X-ray imaging[64], maximum likelihood[70] and Hough Transforms[71].

In the next section it is suggested that the scan statistic is a powerful and general statistic for which good approximations exist for the chance probabilities. It has recently been extended to two dimensions by Loader[72], Chen and Glaz[73] and Alm[30]. Kulldorf[74] has extended this further to higher dimensional searches.

4.2. 2-Dimensional Scan Statistic

This is the two-dimensional development of the Scan Statistic introduced above. It will be introduced using a notion of 'elemental' cells from which a two-dimensional scanning window is constructed. In effect, the scanning window may be moved by discrete steps of the size of the elemental cell. Assume that a two-dimensional square region $R = [0, L] \times [0, L]$ of side L is inspected for occurrences of 'sources'[73]. The region is partitioned into $n \times n$ elemental cells so that the size of a cell $h = L/n$. The contents of each of the n^2 cells are independent.

For $1 \leq i \leq n$ and $1 \leq j \leq n$, define a random variable $Y_{i,j}$ as the number of events in the elemental cell $[(i-1)h, ih] \times [(j-1)h, jh]$. A square box of $m \times m$ small cells is scanned over the whole of region R . There will be ν such boxes, partially dependent if $m \geq 2$, with $\nu = 0, (n-m+1)^2$.

Define

$$S(i_1, i_2) = \sum_{j=i_2}^{i_2+m-1} \sum_{i=i_1}^{i_1+m-1} Y_{i,j}$$

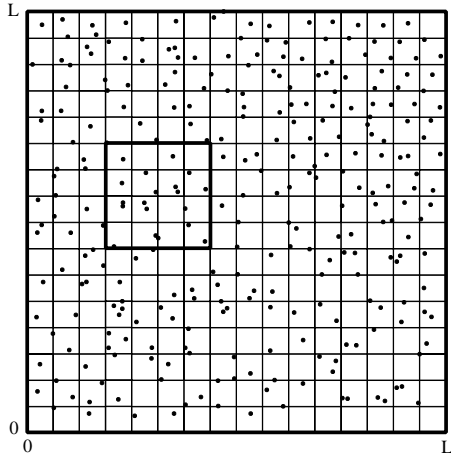


Figure 4. Two-dimensional Scan. The region $[0,L] \times [0,L]$ is partitioned into $n \times n$ cells ($n = 16$) and an $m \times m$ window ($m = 4$) is scanned over it.

to be the number of events in the square box of m^2 adjacent cells starting at $i = i_1$, $j = i_2$. If, during the scanning of the m -box, $S(i_1, i_2)$ exceeds a particular value k , a 'source' has been detected. For $1 \leq i_1, i_2 \leq n - m + 1$ define an 'event' A_{i_1, i_2} as an occurrence of $S(i_1, i_2) \geq k$ and as a member of the set A of all such occurrences.

The two-dimensional scan statistic is defined as:

$$S_m = \max \{S(i_1, i_2); 1 \leq i_1 \leq n - m + 1, 1 \leq i_2 \leq n - m + 1\}$$

and the probability that S_m has at least a value k is:

$$\begin{aligned} P(S_m \geq k) &= P\left(\bigcup_{i_1=1}^{n-m+1} \bigcup_{i_2=1}^{n-m+1} A_{i_1, i_2}\right) \\ &= P\left(\bigcap_{i_1=1}^{n-m+1} \bigcap_{i_2=1}^{n-m+1} A_{i_1, i_2}^c\right) \end{aligned}$$

where A_{i_1, i_2}^c is the occurrence of $S(i_1, i_2) < k$.

4.2.1. Glaz Approximation to 2-D Scan Statistic For a fixed value of $1 \leq i_1 \leq n - m + 1$ the one-dimensional approximation holds:

$$P\left(\bigcap_{i_2=1}^{n-m+1} A_{i_1, i_2}^c\right) \approx q_{2m} \left(\frac{q_{2m}}{q_{2m-1}}\right)^{n-2m}$$

and since $n - m + 1$ square regions of $m \times m$ are scanned, a reasonable approximation is[26]:

$$P(S_m \geq k) \approx 1 - q_{2m-1} \left(\frac{q_{2m}}{q_{2m-1}}\right)^{(n-2m+1)(n-m+1)} \quad (14)$$

For a Poissonian distribution of events the following expression was found to be a good approximation[73]:

$$P(S_m \geq k) \approx 1 - \exp(-\lambda^*) \quad (15)$$

where the approximate mean for the asymptotic Poisson distribution is

$$\lambda^* = 1 - q_{2m-2} + (n - 2m + 2)(n - m + 1)(q_{2m-2} - q_{2m-1})$$

and

$$q_{m+l-1} = P(A_{1,1}^c \cap A_{1,1}^c \dots \cap A_{1,l}^c)$$

Tables are given[73] of this and other approximations, for the Poisson model of $m \leq 20$ and $n \leq 500$.

4.2.2. Alm Approximation to 2-D Scan Statistic A recent paper[30] has given an approximation based on a modification of the method of counting *upcrossings* used in equation 7, which is easy to calculate and moreover is given for a more generally useful rectangular scanning window $[0, a] \times [0, b]$ in a rectangular region $[0, S] \times [0, T]$. The scan statistic L is the maximum content of a scanning window with a two-dimensional Poissonian process X with event density λ :

$$L = L(\lambda, a, b, S, T) = \max_{a \leq t \leq T, b \leq s \leq S} X([t - a, t] \times [s - b, s])$$

The probability of observing at least n events in a scanning window is:

$$P(L \geq n) \approx 1 - F_{N(a)} e^{-\gamma_{n+1}} \quad (16)$$

where

$$\gamma_{n+1} \approx \left(1 - \frac{\lambda ab}{n+1}\right) (T - a) b \lambda (\mu_n - \mu_{n+1}) e^{-\mu_{n+1}}$$

$$\mu_n \approx \left(1 - \frac{\lambda ab}{n}\right) \lambda a (S - b) p_{\lambda ab}(n - 1)$$

and

$$F_{N(a)} \approx F_{\lambda ab} \exp \left[1 - \left(1 - \frac{\lambda ab}{n+1}\right) \lambda a (S - b) p_{\lambda ab}(n) \right]$$

p_μ and F_μ are the Poisson probability and cumulative probability distributions.

The predictions of equation 16 have been compared with the results of 10^5 Monte Carlo simulations in table 3 for $S = T = 20$, $a = b = 2$ and $N = 800$. The agreement is good, allowing for the errors inherent in the Monte Carlo results. For interest, the final column shows the Poissonian probability obtained if the cells were treated as independent. In this particular case, the result of assuming independence of the cells would be a fairly consistent overestimate of the significance of the 'source' by about ' 3σ '. The precise amount of underestimate of the chance probability will depend on the number of elemental cells in the scanning window. Finally, the treatment of [30] has been extended to other shapes of scanning window, such as circular.

4.2.3. Summary In summary, the 2-D Scan Statistic is a preferred general statistic for those cases where events are located randomly on a plane, within fixed bounds, and where there is no *a priori* expectation such as a known source with known instrumental spread function. In most practical situations a good approximation is obtained by using equation 16.

n	Monte Carlo	Equation 16	Poisson
7	0.9990	0.9975	3×10^{-3}
8	0.8658	0.8795	10^{-3}
9	0.4271	0.4501	2×10^{-4}
10	0.1351	0.1346	4×10^{-5}
11	0.0326	0.0304	7×10^{-6}
12	0.0089	0.0059	10^{-6}
13	0.0021	0.0011	2×10^{-7}
14	0.0005	0.0002	3×10^{-8}

Table 3. Comparison of 2-D Scan Statistic probabilities for 100000 samples with $N=800$, $S=T=40$, $a=b=2$. Also shown are the simple Poissonian probabilities assuming independent cells

5. Bayesian Methods

5.1. Introduction

For many workers in cosmic rays, Bayesian methods are relatively novel and the following section attempts to summarise the main ideas and methods. A much fuller development of the ideas discussed below is given by Loredó[75, 58, 76].

5.1.1. Statistics The term 'statistics' arises from the concept of a *statistic*. A statistic is a number derived from observed data and which obeys certain rules, some of which depend on a hypothesis about the system under observation, some of which are extraneous. From this number, one can say how likely it is that the data was drawn from a population obeying rules specified by the particular hypothesis, assuming that all extraneous quantities are allowed for. From that, by an inversion of logic, it is inferred how likely is the hypothesis. In many cases, a particular statistic is used because the experimental results appear to be presented, or may be rearranged to be presented, in a form which allows an easy calculation of that statistic. An example is the epoch-folding of time-tagged photon times above, followed by χ^2 calculated from the binned phases. As was pointed out in that example, some arbitrary choices had to be made which rendered the results unsatisfactory. Also, the aspects of the experimental data which were used to calculate a statistic may not be all that are available, or the most discriminating aspects. This should, with careful design, be evident from a consideration of the statistic's 'power' but not necessarily. It is the claim of Bayesians that such problems are inherent in 'classical' statistics and derive from a misunderstanding of the meaning of Probability.

5.1.2. The Meaning of Probability There were at the beginnings of the subject, and still are, two schools of thought. The first school maintains that the term *probability* is

a statement of the frequency of occurrence of data, such as that taken in a very large number of repeats of an experiment, under the assumption that random factors are at work causing the possibility that the results could be different every time. Take, as an example, coin-tossing: the probability of heads is obviously 0.5 in a single toss. Everyone would agree that, assuming no trickery, an unbiased coin would land equally likely as 'heads' or 'tails'. But there are forces at work which affect the way a coin would land - all amenable to analysis. In fact a coin-tossing machine could be made which obtained 'heads' or 'tails' every time. We regard coin-tossing as a random activity only because we expect humans to apply unconscious variability to the force and direction of the flip which is much greater than that needed in the initial conditions to obtain one more extra turns before landing. This illustrates an important point: unless the hypothesis is clear and specifies all pertinent factors there could be an apparent randomness. That is not to say that true randomness does not exist, only that it is often used as an alibi for lack of knowledge or precision in stating the experimental conditions.

An alternative definition of *probability* is 'a measure of belief in a certain hypothesis'. This is, and was, a much easier idea to grasp but one which was felt from an early date not to be capable of exact or scientific analysis. One consequence of this idea is that for a unique set of data, perhaps taken on a naturally-occurring phenomenon, the idea of a very large set of repeated experiments to plot out the 'frequency distribution necessary to use a 'statistic' was unrealistic. It is this definition which underlies Bayesian thought, and indeed is the definition which more closely accords with the questions for which measurements are made. Interestingly, this meaning of probability explains the frequency version as a special case using de Finetti's representation theorem for exchangeable sequences of events[77].

The main difference between the two philosophical approaches is how the data are related to the hypotheses.

- (i) The 'Frequentist' approach: We obtain $P(D | H)$, the conditional probability of obtaining the observed data, given a particular hypothesis. The hypothesis is frequently a model M which has a parameter space θ and $P(D | M, \theta)$ is the 'sampling distribution' for the data, given the model. A frequently met hypothesis is the 'null' hypothesis H_o in which the parameters are set to zero. For any hypothesis, a statistic is formed which is 'locally most powerful' or even better 'uniformly most powerful' and the probability of observing the data is assigned from a knowledge of the statistic's distribution function. This is now used to give a range of values in which the value of a statistic may fall by chance, with given probability (i.e. frequency). As an interesting aside, it is most usually the case for continuous measures, and frequently for discrete measures, for a range of values of the statistic, including that observed (but also including many values *not* observed), to be used to derive the probability. This is frequently performed by integrating $P(D | M, \theta)$ over the sample (data) space. That is to say, the probability of a hypothesis is determined by the data taken, plus a whole range of values of data which were *not* observed. This curious situation is not often questioned by ordinary users of

'statistics'.

- (ii) The 'Bayesian' approach: We obtain $P(H | D)$, the probability of a hypothesis, given the data - apparently a more difficult matter. However Bayes Theorem gives:

$$\begin{aligned} P(H | D)P(D) &= P(D | H)P(H) \text{ leading to} \\ P(H | D) &= P(D | H)P(H)/P(D) \end{aligned}$$

$P(D | H)$ is the likelihood function, $P(D)$ is the global likelihood, usually treated as an ignorable normalising constant, $P(H)$ is the 'prior' probability of the hypothesis. In addition to the extra terms not used in frequentist analysis, a crucial difference between the approaches is that Bayesian methods would integrate the likelihood function over the parameters space, rather than the sampling (data) space. Bayesian methods have been criticised for the inclusion of an apparently subjective quantity $P(H)$ but a trivial example demonstrates that frequentist analyses are not free from this. Frequentists would determine if the hypothesis H_o of a histogram having all the cells identical were true by taking χ^2 as a statistic. They would use no prior information or knowledge. But we know that histogram cells cannot contain negative numbers, and so some relevant background information is ignored when using χ^2 .

5.1.3. An Example An example of the different approaches is an experiment in which a coin is tossed N times. It lands heads H times. The question is: is it biased? In the Frequentist approach a hypothesis (the 'null' hypothesis) is formed that the coin is unbiased and that the result is a function of randomness only. A sufficiently low probability which is obtained for a suitable statistic would be evidence that the 'null hypothesis' should be abandoned. The binomial distribution describes the result of such discrete, bounded experiments. The probability of H heads in N tosses is $0.5^H \times 0.5^{N-H} \times C_H^N$. One then calculates the probability of obtaining $0, 1, 2, \dots, H-1$ heads and add them to the probability of H heads and say: 'if the null hypothesis is true, getting H or fewer heads in N tosses can occur due to chance, with a given probability. This could be interpreted as some evidence against the 'null hypothesis', hence evidence that the coin is biased.

5.1.4. Stopping Rules Setting aside for the moment the fact that we did not see $0, 1, 2, \dots, H-1$ heads, the last conclusion supposes that the coin was tossed, irrespective of the result, N times and that the number of heads H was the random variable. But suppose the coin was actually tossed by a person until H heads were obtained, and that happened to occur after N tosses. In this case the number of tosses N is the random variable and the number of heads H is fixed. The probability is then derived by combining the individual probabilities of obtaining H heads from $h \geq H$ tosses, and may be significantly different from the first probability calculated.

Loredo[58] gives a more apposite example: a theorist predicts that $f = 10\%$ of the stars in a cluster should be of type **A**. An observer reports 5 stars of type **A** out of 96

observed. The theorist calculates as follows: $N = 96$ and $f = 0.1$ gives 9.6 predicted type **A** stars. The probability of the value of $\chi^2 = 2.45$ of this information is $P = 0.073$, which is acceptable at the 95% level. The observer however decided in advance to stop when he had found 5 stars of type **A**. The expected value of N is then $5/0.1 = 50$ with variance $5(1-f)/f^2 = 450$. The probability of the value of $\chi^2 = 4.7$ of this information is $P = 0.032$, which is *not* acceptable at the 95% level. This ambiguity arises because of the stopping rule used by the experimenter that is - what data sets *might* have been observed.

The stopping rule can therefore be important in classical statistical analysis, and ignorance of the actual rule used may lead to an erroneous or at least ambiguous conclusion. Knowledge of the exact stopping rule is less important in Bayesian analysis, but is valuable in particular when it contains useful information about the unknown quantities. In other cases, the stopping rule could be important if the existence of some data is unknown to the analyser, perhaps because its analysis did not show it to be significant and it was suppressed by the experimenter. The message from this example is that for Frequentist analysis to be possible, an experiment must be precisely defined and if the execution is different in any way from the plan, the data could be worthless.

5.1.5. Conclusion In summary, frequentist methods establish $P(D \mid \theta M)$ as the sampling distribution of the data, given a model M with parameters θ and perform integration over the data space. Bayesian methods start with the same function $P(D \mid \theta M)$ but treat it as a likelihood with integrations performed over the parameter space of the model. In particular, parameters which are necessary for the specification of the model but are not of interest (for example the phase when looking for a periodic signal) are integrated out, or marginalised.

In Bayesian theory, the notion of a 'random variable' is absent so ambiguity does not arise for many types of stopping rule and there is no need for a 'reference set' of hypothetical data. This state of affairs results from the need in Bayesian methods to be specific about all the hypotheses, or to integrate away any unspecifiable variable. Taking again the example of a histogram, and the question of whether its cells are consistent with uniformity using χ^2 . If the null hypothesis is the only hypothesis available, the use of χ^2 is as a 'goodness-of-fit' test for the supposition of flatness. The number observed in the i^{th} bin of n bins is x_i . The number expected in each bin, under the null hypothesis, is $avg = \sum_{i=1}^n x_i/n$ and $\chi^2 = \sum_{i=1}^n (x_i - avg)^2/avg$, assuming avg is large enough (usually 10 or so) for asymptotic normality. The probability of χ^2 for $n - 1$ degrees of freedom is interpreted as supporting or otherwise the null hypothesis. This statistic suffers from a major problem in that it ignores information - the order of the bins may be significant, and so it implicitly assumes a class of alternative models in which the order is unimportant. This can be partially rectified by applying an independent test which is only determined by the order of the bins - the Run Test. This is only applying a patch, since the Run test is most powerful against monotonicity and not other patterns. Frequentists acknowledge this problem in general by using the

idea of the power of a statistic, that is its ability to identify correctly a true model from a particular alternative. Both approaches have subjective factors: Bayesian in assigning prior probabilities to hypotheses, Frequentist in the notion of randomness and its applicability in a mathematical sense to cover for a lack of knowledge of the exact experimental conditions. A consequence of this is that different experts in both fields may come to different conclusions given the same data. Another way of putting this is that the result of analysing data will be a conclusion within a range, depending on (a) the Bayesian priors, or (b) the estimate of the degrees of freedom and unknown factors.

5.2. Bayesian On/Off Analysis

The Bayesian ideas in the above section have recently been applied to the ON/OFF problem treated earlier. As in all Bayesian analyses, some judgement must be made of the priors to be used, but in the cases discussed here the results do not depend critically on how these priors are chosen.

An initial Bayesian analysis of the problem of detecting a source in an ON/OFF counting experiment has been given by Loredó[75]. Using the same notation as in the ON/OFF section above, the probability of the background rate b (the posterior density) from the OFF-source data is:

$$p(b \mid N_{OFF}) = p(N_{OFF} \mid b) \frac{p(b)}{p(N_{OFF})}$$

The Poisson likelihood for N_{OFF} is:

$$p(N_{OFF} \mid b) = \frac{(bT)^{N_{OFF}} e^{-bT}}{N_{OFF}!}$$

The parameter b is unknown and so the 'prior' probability would appear to be a matter of guesswork. If the range of b were pre-specified in some non-arbitrary way, at least the scale of b would be known, and a flat prior would be reasonable. If even the scale of b is unknown, the 'least informative' prior for b is $p(b) = 1/b$, which is uniform in $\log b$, and then

$$p(N_{OFF}) = \frac{T^{N_{OFF}}}{N_{OFF}!} \int_0^\infty b^{N_{OFF}-1} e^{-bT} db$$

This leads to

$$p(b \mid N_{OFF}) = \frac{T_{OFF} (bT_{OFF})^{N_{OFF}-1} e^{bT_{OFF}}}{(N_{OFF} - 1)!}$$

Note that the expectation of the background $\hat{b} = N_{OFF}/T$ and that the assumption of $p(b) = 1/b$ does not strongly affect the result, Loredó pointing out that a prior uniform in b only marginally alters the expectation $\hat{b} = (N_{OFF} + 1)/T$. The joint probability of the background rate b and a source rate s , given N_{ON} and N_{OFF} , is:

$$p(sb \mid N_{ON}) = p(s \mid b) p(b) \frac{p(N_{ON} \mid sb)}{p(N_{ON})}$$

The probability of the source rate s is obtained by marginalising b , that is $p(s | N_{ON}) = \int p(sb | N_{ON}) db$:

$$p(s | N_{ON}) = \sum_{i=1}^{N_{ON}} C_i \frac{T_{ON} (sT_{ON})^{i-1} e^{-sT_{ON}}}{(i-1)!} \quad (17)$$

where

$$C_i = \frac{\left(1 + \frac{1}{\alpha}\right)^i \frac{(N_{ON} + N_{OFF} - i - 1)!}{(N_{ON} - i)!}}{\sum_{j=1}^{N_{ON}} \left(1 + \frac{1}{\alpha}\right)^j \frac{(N_{ON} + N_{OFF} - j - 1)!}{(N_{ON} - j)!}}$$

This result is formally correct for all positive values of N_{ON} and N_{OFF} and is particularly useful for small values when the asymptotic treatments fail.

Its main value is to illustrate the completely different approach and result of the application of Bayesian ideas. However, there are some computational problems for values of N_{ON} and N_{OFF} which exceed ~ 100 . For values of N_{ON} and N_{OFF} which are less than ~ 100 evaluation of equation 2 and equation 17 shows small differences in the derived probabilities.

5.3. Bayesian Change Point Analysis - Bursts

A recent paper by Scargle[8] has used Bayesian methods to analyse structure in photon counting data. It is worth noting that the ON/OFF problem dealt with above is a special case of change point analysis, where there is only one change point and its location is known in advance. The principles are the same as those outlined above, with the added simplicity of having simpler alternatives to the uniform model. The uniform counting rate model M_1 assumes a constant intensity over a particular time interval T . An alternative model M_2 has the interval T broken into two regions T_1 and T_2 , $T = T_1 + T_2$, each with a different counting rate. In general, a model M_k may be constructed with k regions. Bayes Theorem give the probability of a model

$$p(M_k | D, I) = \frac{p(D | M_k, I) p(M_k | I)}{p(D | I)}$$

Dropping the explicit appearance of the background information I , the odds ratio O_{kj} between two competing models M_k and M_j is then

$$\frac{p(M_k | D)}{p(M_j | D)} = \frac{p(D | M_k) p(M_k)}{p(D | M_j) p(M_j)}$$

The parameter θ or vector of parameters $\vec{\theta}$ of the model M_k enter when $p(D | M_k)$ is calculated

$$p(D | M_k) = \int p(D | \vec{\theta}, M_k) p(\vec{\theta} | M_k) d\vec{\theta}$$

The odds ratio O_{kj} is then

$$O_{kj} = \frac{p(M_k | D)}{p(M_j | D)}$$

$$\begin{aligned}
&= \frac{p(M_k) \int p(D | \vec{\theta}, M_k) p(\vec{\theta} | M_k) d\vec{\theta}}{p(M_j) \int p(D | \vec{\theta}, M_j) p(\vec{\theta} | M_j) d\vec{\theta}} \\
&= \frac{p(M_k) \mathcal{L}(M_k, D)}{p(M_j) \mathcal{L}(M_j, D)}
\end{aligned} \tag{18}$$

where $\mathcal{L}(M_k, D)$ is the global likelihood of model M_k .

For the constant-rate model M_1 , N events arrive in a time T which is treated as being divided into M intervals of duration δt , the justification being that photon counting apparatus always has a resolving time. Note that the number of events in any particular interval δt can be 0 or 1 only. The author shows that the global likelihood for this constant-rate model of Time Tagged Events (TTE) is

$$\mathcal{L}(M_1 | TTE) = \frac{\Gamma(N+1)\Gamma(M-N+1)}{\Gamma(M+2)}$$

If the data is time-binned into M equal bins, but such that any number of events may occur in any bin, given an overall rate $\lambda = N/T$ and mean number per bin of $\mu = \lambda T/M$, the global likelihood is:

$$\mathcal{L}(M_1 | Binned) = \frac{\Gamma(N+1)}{(M+1)^{N+1}}$$

Note that the bins are fixed and may not be scanned to maximise \mathcal{L} .

The alternative model M_k has a likelihood which is the product of the likelihoods of the individual constant-rate regions of T . For a two-rate model with the time of the change of rate being t_{cp}

$$\begin{aligned}
\mathcal{L}(M_2 | D) &= \int dt_{cp} \int d\Lambda_1 \int d\Lambda_2 p_{cp}(t_{cp}) \times p[D_1 | M_1(\Lambda_1, T_1)] p(\Lambda_1) \\
&\quad \times p[D_2 | M_2(\Lambda_2, T_2)] p(\Lambda_2)
\end{aligned}$$

where $\Lambda = \lambda \delta t$, $P(\Lambda)$ is the prior for the rate Λ and P_{cp} is the prior for the change-point time t_{cp} . For time-tagged data with resolution δt the integrals are sums and the change-point location is $m_{cp} \delta t$. Since the change-point can be tested only at the arrival time of a photon, the photon number of the change-point n_{cp} is used as an index. The number of events in the first section, up to the change-point, is $N_1 = n_{cp}$, $N_2 = N - N_1$ and $M_1 = m_{n_{cp}}$. The global likelihood is then

$$\begin{aligned}
\mathcal{L}(M_2 | D) &= \sum_{n_{cp}} \frac{\Gamma(n_{cp}+1)\Gamma(m_{n_{cp}} - n_{cp} + 1)}{\Gamma(n_{cp}+2)} \\
&\quad \times \frac{\Gamma(N - n_{cp} + 1)\Gamma(m_{N-n_{cp}} - (N - n_{cp}) + 1)}{\Gamma(N - n_{cp} + 2)} \Delta t_{n_{cp}}
\end{aligned}$$

The paper[8] gives a coding in a popular mathematical package to implement the above ideas.

5.4. Bayesian Periodicity Analysis

5.4.1. Introduction Frequentist statistical theory allows more than one test to be applied to any situation. Any statistic, or function of the data, may be defined and

the 'best' is selected depending on its 'power' or likelihood of selecting the 'correct' hypothesis. One of the problems of the frequentist approach to looking for evidence of periodicity is that, in the absence of a specific light curve, the alternative hypothesis (to one of uniformity in the phase distribution) is unknown and the power of a statistical test is difficult to specify except for a narrow class of alternative light curves. The Rayleigh statistic, Z_1^2 , is powerful only for the fundamental period and is formally the most powerful test for alternatives to uniformity from the Von Mises distribution - the circular equivalent of the Gaussian on the line. The Z_n^2 test allows the addition of $n - 1$ harmonics but needs a protocol to decide when to stop adding harmonics and therefore degrees of freedom (the H test mentioned above suggests such a protocol). Finally, Protheroe's test is powerful for very narrow light curves. Each could be tried in succession to look for evidence of periodicity, but a method which is indifferent to the shape of the light curve, without any penalty, would be of great advantage.

5.4.2. Gregory & Loredo Method Such a method based on Bayesian analysis, is claimed by Gregory and Loredo[78]. The essence of the method is to compare a uniform model for the distribution in phase at a trial frequency with a periodic model. The great difference between this and other methods is how the periodic model is proposed and how the necessary uncertainties and their associated 'degrees of freedom' of classical theory are accounted for. In particular, since an arbitrary postulated light curve may be of any shape, the method automatically applies Ockham's razor, in that models with fewer variables are automatically favoured unless the evidence from the data more than compensates. More complicated light curves (not necessarily with small number of harmonics, a δ -function is uncomplicated in this context) are penalized for their greater complexity.

Bayes Theorem is used to compare the probabilities of two parameterised models of the phase distribution. In the notations of the authors, the probability that a model M describes the data, given the data D and any background information I is

$$p(M | D, I) = p(M | I) \frac{p(D | M, I)}{p(D | I)} \quad (19)$$

The first term on the right, $p(M | I)$, is the prior probability of the model M , which may seem to be subjective but may be estimated in some cases from the permissible range of the parameters. The numerator in the second term, $p(D | M, I)$, is the sampling probability of the data D , or the likelihood of the model M . The denominator, $p(D | I)$, is the global likelihood of the entire class of models. If the model contains a parameter θ , or in the case two or more parameters a vector $\vec{\theta}$, the likelihood of the model can be calculated:

$$p(D | M) = \int_{\vec{\theta}} p(D | \vec{\theta}, M) p(\vec{\theta} | M)$$

For time-tagged photon data with N events detected over a time T , the probability of D for a particular rate model $r(t)$ can be calculated. For the time T divided into

very small intervals of length Δt , the probability of n events in Δt is:

$$p_n = \frac{[r(t)\Delta t]^n e^{-r(t)\Delta t}}{n!}$$

If Δt is small enough for $p_i = 0, i \geq 2$ then the sequence of $T/\Delta t$ time samples will contain N containing one event and $Q = T/\Delta t - N$ containing no event. The likelihood is then:

$$p(D | r, I) = \prod_{i=1}^N p_1(t_i) \prod_{k=1}^Q p_0(t_k)$$

Using $p_0(t) = e^{-r(t)\Delta t}$ and $p_1(t) = r(t)\Delta t e^{-r(t)\Delta t}$ the likelihood function is

$$p(D | r, I) = \Delta t^N \left[\prod_{i=1}^N r(t_i) \right] \exp \left[- \sum_{k=1}^{N+Q} r(t_k) \Delta t \right]$$

In the case of a periodic model, the non-uniformity in phase is characterised by the varying contents of the phase bins. Although the number of phase bins needed to detect any light curve and the origin of phase are unknowns, these will be marginalised or integrated out. If there are m phase bins the average rate $A = \frac{1}{m} \sum_{j=1}^m r_j$ and the fraction of the total rate per period in phase bin j is $f_j = \frac{r_j}{mA}$. The likelihood function is shown to reduce to

$$P(D | \omega, \phi, A, \mathbf{f}, M_m) = \Delta t^N (mA)^N e^{-AT} \left(\prod_{j=1}^m f_j^{n_j} \right)$$

where ω is the postulated angular frequency, ϕ the starting phase, \mathbf{f} the set of m values of f_j and n_j being the number of events occurring in bin j .

The joint prior density for the parameters $\omega, \phi, A, \mathbf{f}$ is

$$p(\omega, \phi, A, \mathbf{f} | M_m) = p(\omega | M_m) p(\phi | M_m) p(A | M_m) p(\mathbf{f} | M_m)$$

The prior densities are:

- (i) $p(\phi | M_m) = 1/2\pi$, this assumes that any starting phase is equally likely,
- (ii) $p(A | M_m) = 1/A_{max}$, this assumes that A does not change during the observation and any value of A from $A = 0$ to $A = A_{max}$ is possible,
- (iii) $p(\omega | M_m) = \omega \ln(\omega_{hi}/\omega_{lo})$, where $[\omega_{hi}, \omega_{lo}]$ is a prior range for ω ,
- (iv) $p(\mathbf{f}) = (m-1)! \delta \left(1 - \sum_{j=1}^m f_j \right)$.

The assignment of the priors of the models themselves is all that is needed before comparing the likelihoods of the models. The two models are equally likely *a priori* and so the prior likelihood of the non-periodic model (M_1), $p(M_1) = 1/2$ and that for the periodic model ($M_m, m = 2, m_{max}$), $p(M_m | I) = 1/2\nu$ where $\nu = m_{max} - 1$.

The final result for the odds O against a uniform model of phase and in favour of a periodic model with phase and period unknown (a common case) is:

$$O = \frac{1}{2\pi\nu \ln(\omega_{hi}/\omega_{lo})} \frac{N!(m-1)!}{(N+m-1)!} \int_{\omega_{lo}}^{\omega_{hi}} \frac{d\omega}{\omega} \int_0^{2\pi} d\phi \frac{m^N}{W_m(\omega, \phi)} \quad (20)$$

where $W_m(\omega, \phi)$ is the number of ways that the set of n_j observed counts can be made by distributing N counts in m bins:

$$W_m(\omega, \phi) = \frac{N!}{\prod_{j=1}^m n_j!}$$

and n_j , the number of events placed in the j^{th} phase bin depends on ω , ϕ and m .

If the period is known, this reduces to:

$$O(\omega) = \frac{1}{2\pi\nu} \frac{N!(m-1)!}{(N+m-1)!} \int_0^{2\pi} d\phi \frac{m^N}{W_m(\omega, \phi)} \quad (21)$$

In order to illustrate the difference between the information available from this treatment and from the Rayleigh test, a data set has been generated containing time-tagged random events with a constant mean rate, plus a periodic component. The results are shown in figure 5.4.2. A particular point to note is that although the Rayleigh power is always positive, even for pure noise, in the case of LOG(Bayesian odds), peaks do not become 'interesting' until they become positive. This is because the 'degrees of freedom' have been accounted for automatically and cause the offset seen in figure 5.4.2 so that peaks falling below $\log(Odds) = 0$ are just those expected from noise. It can be seen much more clearly in the Bayesian Odds diagram that there is only one significant peak, at the period simulated.

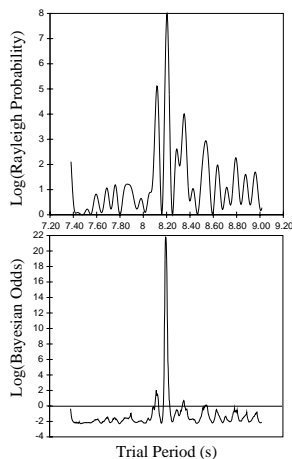


Figure 5. A comparison of Rayleigh Power and Bayesian Odds using random simulated data with a periodic signal at $P = 8.2s$

The method has been used to detect a weak pulsar signal from SNR 0540-693 in ROSAT data, which could not be detected using a standard FFT technique[79]. Moreover, the precision of determining the frequency was much higher for the Bayesian method than for χ^2 using epoch-folding. The frequency precision of the latter is determined mainly by the duration of the data and is not strongly influenced by the number of photons. Gregory and Loredo show[79] that the Bayesian method obtains greater precision in parameter estimation with more photons. The method has also been used to detect 1600 day modulation in the long-term radio emission of an X-ray binary,

with very non-uniformly sampled data and a Gaussian noise of unknown magnitude [80, 81]. There has been a recent independent use of a Bayesian method to calculate the upper limit to a pulsed flux at a known period, independent of pulse width and pulse phase[82].

6. Conclusions

The hope of this review is that the more commonly met data analysis problems may be approached by the cosmic ray worker with a more consistent and up to date approach. There have been a number of advances in recent years in the tools, and more importantly in the methods, available to cosmic ray experimenters to ensure that the maximum use is made of hard-won data. The traditional statistical methods have resulted in a measure of agreement on the 'correct' way to look for sources from ON/OFF data, change points (bursts) in 1- and 2-dimensions and in periodicity. The application of these methods requires care to ensure that the 'degrees of freedom' are kept under control and properly accounted for: many of the criticisms of claimed sources have been based on the latter.

New Bayesian methods of testing hypotheses have recently been proposed. A central theme of these methods is that classical methods often cloak ignorance in a way which distorts the results. There are claimed to be significant benefits to the use of Bayesian methods which derive from the requirement to be absolutely specific about the hypotheses and the methodology of marginalising nuisance parameters. In contrast to classical statistical methods, where various *statistics* may be generated from the same data, each with different assumptions, *degrees of freedom* and power, Bayesian methods provide a framework for describing completely the data and allow the direct comparison of specified hypotheses. A practical result of the philosophical differences between the approaches is that, rather than relying on a relatively easy-to-use, pre-packaged test statistic, with the accompanying dangers of hidden degrees of freedom, a Bayesian method requires the data interpreter to model the hypotheses precisely. The obvious disadvantages of this are claimed to be more than compensated by the directness of the link between the hypotheses and the data. Bayesian methods may require some time to become accepted in the field, in that the methodologies and ideas have not traditionally been part of the training of physicists; indeed may not have been as useful if physicists' training in classical methods had been better.

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